

Revealing the secrets of the number pi and copper number algorithms

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Abstract

The present text constitutes a compilation of humanity's fervour to ascertain the enigmatic number pi, entitled "Revealing the secrets of the number pi and the algorithms of the copper number". In this essay, we seek to illuminate the hidden intricacies of this number, explore its potential connections with other numerical entities, investigate the alleged falsehoods associated with its proponents, and delve into more complex subjects by employing calculus, series, and contemporary technologies. The document's intention is to demonstrate the methods and justify the interesting procedures that lead to the solution of numerical problems, with a particular focus on the number pi. It is a valuable resource for mathematicians, computer scientists, and anyone with an interest in formulas or the human ingenuity behind their discovery. The author posits that each contribution of knowledge leads to a subsequent step, and the discovery of the so-called "absolutely convergent series" will allow the reader to devise an algorithm to find all the digits of pi.

Introduction

We provide the reader with basic instructions to deepen the topics of classical geometry, use of freely available computational tools and the background of the most relevant concepts of numerical calculus such as series and limits, which will allow us to contextualize theoretical foundations to unravel the enigmatic number pi. From the perspective of mathematical authors, in history, we can perceive the longing and inspiration that lead to the discovery of an emblematic problem defined by Archimedes as “The squaring of the circle” and later named by Euler with the Greek letter (pi) (π). A problem that existed long before and that apparently in all ages unified wise men in a common purpose. All these authors wrote about it, committed themselves to obtaining it, motivated by being the first to see what others had not been able to, and then, in the face of failure and resignation, they gave way to future generations.

One of the most important numbers in mathematics is the number pi, probably not a common or necessary number for many people. However, this number is present in a large number of equations and is part of the solution of many scientific statements. Some important examples, it is present in Albert Einstein's famous equations of general relativity, in the cosmological constant of the gravitational curvature tensor $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$. It is also present in the Heisenberg uncertainty principle, which introduces a fundamental limit of particle theory and where the energy of a photon with angular frequency $\omega = 2\pi f$, can be expressed as $E = \hbar\omega$, which is one of the most important formulas of quantum mechanics [8]. The number pi is found in many other formulas and equations, in probability theory, in more tangible matters, perhaps, such as in the geometry of space, and it is because of this that some scientists distinguish it with a metallic name of “copper”, since it is in many cases the essence of things.

The challenge associated with the number pi is to find more and more of its decimals, since it is irrational, transcendent and to get to it will have to pass first through infinite limits and Euler's e number. Obtaining its digits became a kind of competition among scientists and computer scientists in the world. The current record is held by a data storage company called Solidigm [10], who announced it in March of this year 2024, before 2023 the

record belonged to Google Cloud. The company announced on March 15 (date that celebrates the day of the number pi within the year) that they had managed to reach the mark of 105 trillion digits of this number, but this changes very fast and it seems to me that already in August this year is around 300 trillion decimals discovered, using computational methods, which demands large amount of memory and large amount of processes. This may be crazy, but Akira Haraguchi, a Japanese engineer and mental therapist, is known for breaking the world record for memorizing digits of the number pi, which he managed to remember 100,000 digits and demonstrated it on October 3, 2006. Akira started reciting the decimals of pi in the plenary hall of the city hall of Kisarazu, Japan, and it took him a total of 16.5 hours to recite all the digits he had memorized without making any mistakes [9]. Could this be another obsession of the human fervor for discovery? What could be obtained if we achieve more decimals of pi, some believe that something can happen, and the fact of being able to work with infinitely large numbers such as Graham's number or others incredibly accurate as pi, could perhaps open doors to unexplored areas of knowledge.

In this essay, we will develop background issues that lead to obtaining these decimals of pi and knowing that computers are capable of so much more, we will be able to see the digits of pi and help Akira recite the decimals to count from 100,001 onwards. We will explain motivations, use tools and give an account of some secrets, which are kept in patents of those who have made this work a professional activity. We will explain the most famous methods and describe how to arrive at a new algorithm, of our authorship in Python, which we will call Algorithm ASS.

The devastating problem of squaring a circle

It is imperative to note that in order to approach geometric structures, the utilization of Geogebra is strongly recommended.

Throughout history, mathematicians and geometers, whether amateurs or seasoned professionals, have consistently endeavored to explore these concepts.

The objective is to derive a square with an area equivalent to that of a circle through a finite mechanical sequence of steps. In antiquity, this process entailed the repeated use of a compass, transforming a square into a circle and vice versa.

The crux of the challenge lies in the intractability of obtaining a segment of length equal to the irrational number pi, approximately $\pi \approx 3.141592653589794$. However, it is possible to obtain a segment that measures an irrational number of a similar nature, such as the numbers:

The graphical relationship of the same area, expressed in a square and a circumference, is the same for $\sqrt{2}$ (the diagonal of a square with a side length of 1) or $\sqrt{5}$ (the hypotenuse of a triangle with legs of lengths 2 and 1), or in general for $\sqrt{2}$. This revelation, however, was previously unknown, and throughout history, the same problem persisted, prompting mathematicians to engage with it repeatedly. Perhaps because this kind of problem did not seem too complex for a somewhat experienced geometrician, many claimed its solution, like the root of 2, was possible.

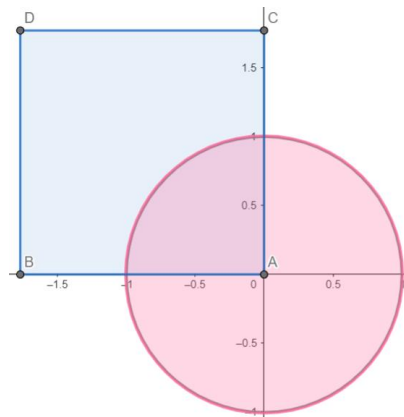


Figure 1. Graphic relationship of the same area, expressed in a square and a circumference.

Part 1. What you need to know before knowing pi

This part is intended to investigate foundational aspects of mathematical thinking and to recognize geometric and mathematical representations that establish numerical regularities. Processes that occurred in the history of mankind and are closely related to the number pi.

1. Perimeter and area of a circle

The Egyptians are credited with many of the primary geometric and mathematical concepts that the Greeks used in their writings. It is evident that Egyptian, Mesopotamian, and Indian mathematics inspired Greeks such as Thales of Miletus (between 624 B.C. - 546 B.C.) and Pythagoras (582 B.C. - 507 B.C.), who gave the initial start to Greek mathematics, or Hellenic mathematics, after the time of Alexander the Great and before the Roman Empire, around 27 B.C.

The Egyptians, using an approximation of the number pi, made their calculations evident when they tried to roll a circle with a drop of ink on papyrus, with which they had an excellent approximation of the perimeter $2\pi r$, applying proportions showed regularities that showed signs of possessing a secret factor implicit in the operation with 2π , which was related to the perimeter of the circle of radius 1.

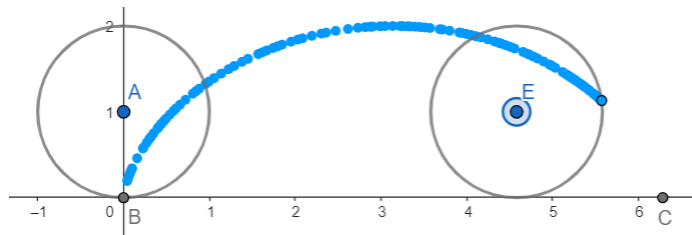


Figure 2. Rotational displacement of a wheel starts from a mark until its circumference is reached.

The Rhind Papyrus (or Ahmes Papyrus, its author dates from about 1650 B.C.) to how to find the value of π by approximating the area of a square with sides 4 and then 8 to that of a circle with diameter 4 and then 8. The area of the circle is approximately equal to the area of an octagon (irregular) of 12 and 24 sides. More details are in the publication of [18] Beckmann P. (1971) Publishes A History of Pi. St. Martin's Press.

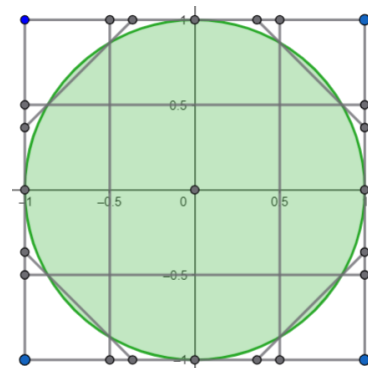


Figure 3. Circle inscribed in a Square. The Square is divided to establish an approximate measurement of each side of the irregular polygon, approximating the perimeter.

$$3 + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} = \frac{256}{81} \approx 3.16$$

2. Archimedean Polygons

It was Archimedes (Archimedes of Syracuse ac. 287 - ac. 212, with an astronomer father) who formalized the problem and managed to have a good approximation of the area of a circumference. He also managed to do many other things, he was a renowned mathematician of the time and was very advanced with his ability to perform physical, mechanical, hydrostatic and other applications, in almost all areas of science. In particular, when he needed the area or perimeter of a circle, he knew that a constant was implicit here, since when modifying the size of the radius in the formulas

of the area $Pi \cdot r^2$ and the perimeter $Pi \cdot 2r$ (of his own authorship), it was only necessary to multiply by this constant Pi . Then he proposed to approximate the perimeter of a circumference of radius equal to 1 (where the unit could be the customary one in Greek times, **1 palame**, which was the width of the palm of the hand without the thumb) and he was aware that then he only had to divide it in half. He considered from the center the 360 degrees of the complete angle, dividing it into equal parts depending on the number of sides of an inscribed polygon and another circumscribed to the circumference. He used trigonometry on each concentric triangle to obtain the lengths of the sides of each polygon and increased the sides of each polygon to "enclose" or bound the value of the perimeter length measure [5].

Thus, depending on the number of sides of the polygon, the cosine of the angle is determined. This task was undertaken by Archimedes with a polygon of 96 sides. Let us recall that $r=1$, and $\frac{360}{96} = \frac{15}{4} = 3^\circ 45' = 3.75^\circ$ [2]

$$2\pi \approx 96 \cdot \sqrt{2 - 2 \cos(3.75^\circ)} \approx 6.282 = \frac{3141}{500} \quad \text{So: } \pi \approx \frac{3141}{1000} = 3.141$$

He succeeded in bounding pi from below with the inscribed polygon [6], and then from above with the circumscribed polygons of 96 sides, thereby obtaining bounds for the value of pi.:

$$3.141 < \pi < 3.142$$

$\alpha = 360/N$ central angle of each concentric triangle starting from one of the sides of the N-sided polygon.

Lc will be the side of the circumscribed polygon.

Li will be the side of the inscribed polygon.

r will be the radius of the circumference, which will be 1.

$N = 6$ is the case of the figure on the right. Then:

$$\frac{Li/2}{r} = \sin\left(\frac{\alpha}{2}\right) \quad y \quad \frac{Lc/2}{r} = \tan\left(\frac{\alpha}{2}\right)$$

If we look only at the situation of the inscribed polygon, it is deduced by trigonometric identity of the mean angle [1]:

$$Li = 2r \sin\left(\frac{\alpha}{2}\right) = 2r \sqrt{\frac{1 - \cos(\alpha)}{2}} = r\sqrt{2 - 2 \cos \alpha}$$

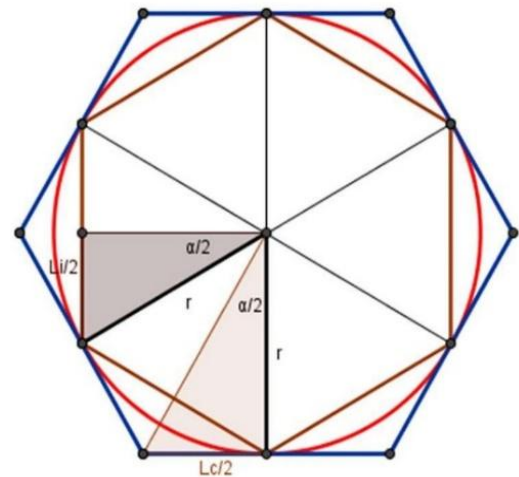


Figure 4. By regular polygons inscribed and circumscribed to the circumference of radius 1.

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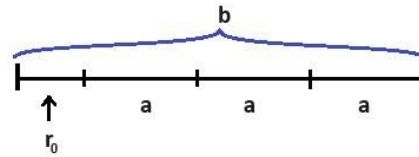
He was able to bound below pi with the inscribed polygon [6] and then above the value with circumscribed 96-sided polygons, obtaining dimensions for the value of pi:

$$3.141 < \pi < 3.142$$

3. Fractional representation of real numbers by Euclid's algorithm

Approximation of a decimal.

Let a, b be any positive integers, with $a \neq b$ and $a < b$. What we want is to obtain an approximation of the fraction $\frac{a}{b}$. By the division algorithm we know that there always exists $q_0 \in \mathbb{Z}$ such that we can write: $b = aq_0 + r_0$ with $0 \leq r < a$. Graphically, the situation is:

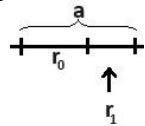


For the case in which $q = 3$

also exists $q_1 \in \mathbb{Z}$ and r_1 such that:

$$a = r_0 q_1 + r_1 \text{ with } 0 \leq r_1 < r_0$$

This is interpreted with the graph:



For the case in which $q_1 = 1$

Thus, we summarize that:

$$\frac{a}{b} = \frac{a}{aq_0 + r_0} = \frac{1}{\frac{aq_0 + r_0}{a}} = \frac{1}{q_0 + \frac{r_0}{a}} = \frac{1}{q_0 + \frac{r_0}{r_0 q_1 + r_1}} = \frac{1}{q_0 + \frac{1}{\frac{r_0 q_1 + r_1}{r_0}}} = \frac{1}{q_0 + \frac{1}{q_1 + \frac{r_1}{r_0}}}$$

As you can see, a convenient stepwise decomposition occurs. Now if the purpose is to have as good an approximation as we want and we will get to one of the finites $\frac{r_{n+1}}{n}$, we will replace them with a value like 1.

For example, let's obtain by this algorithm a reasonable fractional approximation for the decimal number 0.5555 and use all the decimals that the calculator gives us. We start by first asking how many times the unit can be contained in 0.5555 and since it is not, then: $0.5555 \approx 0 + \frac{1}{b}$. We decompose b asking ourselves how many times the first remainder 0.5555 is contained in the unit, from the question $\frac{1}{0.5555} = 1.80018 \dots$, the answer is 1, and we say: $0.5555 \approx \frac{1}{1 + \frac{1}{c}}$. We continue decomposing c with

remainder $r_1 = 1 - 0.5555 = 0.4445$, and ask ourselves how many times is contained in 0.5555, the operation is $\frac{0.5555}{0.4445} = 1.24971 \dots$, and the answer is 1 again. We already have: $0.5555 \approx \frac{1}{1 + \frac{1}{1 + \frac{1}{d}}}$. Now,

we continue with the next remainder $r_2 = 0.5555 - 0.4445 = 0.111$

And we answer how many times it is contained in 0.4445, with the operation $\frac{0.4445}{0.111} = 4.0045$, the answer is 4.

If we leave the algorithmic approach up to this point, we obtain: $0.5555 \approx \frac{1}{1 + \frac{1}{1 + \frac{1}{4}}} = \frac{5}{9}$.

You can confirm that this fractional approach is of very good quality.

Decimal approximation of the root of 2. Let's try with another number, let's get a fractional approximation of the number $\sqrt{2}$, which we know is irrational and use Geogebra to recognize with the compass when one segment is contained in the other [4].

Let's start with a segment that measures this value I am referring to the diagonal of the square, of side one unit. You can verify by the Pythagorean Theorem this statement, being the hypotenuse of one of the halves of the square and being right triangles, the measure of the diagonal is exactly $\sqrt{2}$. Now we can have a geometric representation, using a compass. We start with a circle with center at the lower left vertex and radius 1, it responds to the idea that how many times the side of the square is contained in its diagonal, it digs only once. The excess of the diagonal (outside this first circle) ED, is taken up by the radius of another circle, which, with another of equal size, indicates that this excess is contained twice on the other part of the diagonal FE and GF. Continuing, AG is contained twice over the other component.

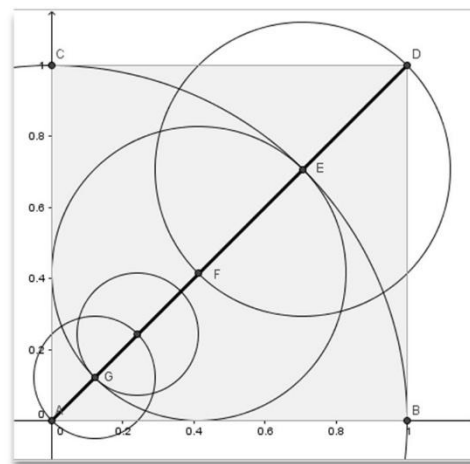


Figure 5. Subdivision of the diagonal of a square with the Geogebra digital compass.

This process is recorded as follows, the times that one segment is contained over the other is in dark color:

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{d_3}}} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{d_6}}}}}} = \frac{99}{70} = 1.414285714$$

As you can see this compass procedure is exactly Euclid's algorithm. Thus, we obtain a reduced fraction, which has a very good approximation to the fourth decimal place. $\sqrt{2} = 1.414213562...$

Euclid's algorithm allows us to obtain the decimals with the desired precision, as it is a nomenclature, modern science calls it "continuous fractional form" and has a presentation format, which in this case is $[1; \bar{2}]$.

Some scholars took this very seriously and you can see how an amateur mathematician defined a method that repeatedly included the root of 2, with which he managed to calculate the exact perimeter of a circumference, see [7].

4. Hippocrates lunula area

While the central problem of the "squaring of the circle" remained unsolved more than 240 years before Archimedes, other equally important issues were still around, and the news of the demonstration of the

lunula of Hippocrates [Greek mathematician Hippocrates of Chios, 470 BC] spreads. This argumentation came to create hope also in the squaring of areas of a part of the circumference, which could serve to continue the work with the entire circumference.

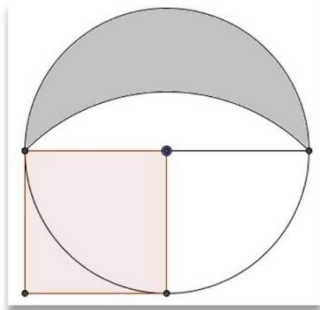


Figure 6. Square of the Hippocrates lunula. The area of the square and the lunula are equal.

The figure on the left corresponds to the squaring of the lunula, a fact that highlights the fervor of this type of theme at the time.

These demonstrations were made with operations similar to the famous Pythagorean Theorem (published around 510 BC), by which the Pythagorean school was created in Magna Graecia, a school of thought that combined philosophical, mystical and scientific aspects. It is that the Pythagorean Theorem was judged by history and it seems to have been discovered much earlier, the Babylonians already used it.

Later, in another part of the world, the Arab mathematician Alhacen (945-1040 AD), recovered the same statement from the Lunulas. Perhaps he found the source, but at that time traveling was very difficult, the publications were handmade and possibly the temptation to be the author of these subjects in another region allowed to be valued among his peers, all limited to know the source. He tried with these arguments of the lunulae to square the circumference and admitted to having failed. Whatever the outcome, the lesson prompted others to rethink these issues. Alhacen had more virtues, he was a precursor of other trends in non-Euclidean geometry by exploring and proposing to modify the "5th Euclidean postulate of parallels" (the fifth postulate of Euclid's Elements).

5.The Fibonacci series

Leonardo of Pisa, better known as Fibonacci (1170-1250 AD), in 1202 AD wrote his book "Liber abbaci" which means "the book of calculation". He explains in his publication about a sequence of the breeding and reproduction of rabbits, which in a few rules could estimate the number of rabbits that there would be in a certain time. The series is generated by the reproductive characteristics of the rabbits in an idealized environment, so that from a pair of rabbits they only reach reproductive maturity after three months. Thus, in the third month they have a pair of offspring (another pair of different sexes), and when these reach maturity in the third month they will be able to add another pair and so it would agree with the sequence counting the pairs of rabbits. Of course, rabbits are not born this way, but that does not matter as long as we describe a special phenomenon and as if things were this way. Fibonacci poses in his book the problem: how many pairs of rabbits are there after one year (12 months)?

Following the series:

Fibonacci Series. This is the count of pairs of rabbits that would be in each month

1	1	2	3	5	8	13	21	34	55	89	144
1	2	3	4	5	6	7	8	9	10	11	12

Answer: 144 pairs of rabbits completed the year. The curious thing about the exercise is that in order to know how many pairs of rabbits there will be next month, it consists of adding those of the two

previous months. That is to say, at the 12th month 144 pairs were completed, when at the 10th and 11th month there were 55 and 89 respectively, it is confirmed $55+89=144$.

This series was not exactly invented by Fibonacci, but he made it very popular in his book, rather all the rest of his work was devoted to making popular other Arabic methods of calculation that we all know today. It happens that Fibonacci's example became more and more known, because some followers rescued his legacy and collected different examples from nature, as if this series was a very probable sequence of the growth in various natural developments: the petals of flowers, stems of plants and leaves of cobs, etc. What is more, this seemed to relate strongly to another number in vogue in antiquity, a special number used by the architects of the time and referred to as the golden number, namely the number ϕ .

$$\phi = \frac{1 + \sqrt{5}}{2}$$

$$= 1,6180339887498948482045868343656381177203091798057628621354486227$$

Interestingly, an incredible relationship was noticed between ϕ and the convergence of a limit that included the Fibonacci series: $\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \phi$ That is, if we divide two consecutive numbers of the Fibonacci series and do it *¡es el 144!*

progressively, we obtain the limit that is the golden number

$$\frac{F_5}{F_4} = \frac{5}{3} = 1.66\ldots \quad ; \quad \frac{F_6}{F_5} = \frac{8}{5} = 1.6 \quad ; \quad \frac{F_7}{F_6} = \frac{13}{8} = 1.625$$

It is anecdotal how does one get to the other? It was one more of these discoveries and the appeal of mathematics. The relationship between the Fibonacci sequence with the number ϕ would also allow us to find any number in the sequence. It can be shown that, for a large number in the position of the Fibonacci series, the relationship is established:

$$F_n \approx \frac{1}{\sqrt{5}} \phi^n \text{ This is deduced from Binet's formula (explained below).}$$

For example, what number will be in position 12 of the Fibonacci series, which we already know, but we will be able to confirm it..., it is calculated: $\frac{1}{\sqrt{5}} (1.618)^{12} \approx 143.96$ and we say then, when rounding, *¡es el 144!* As you can confirm in the rule above, this is true!

6.The gold number, the metallic numbers and out of this category the numbers: platinum and copper

The golden number is the category that receives number ϕ (also number of God). It was used by a Greek sculptor named Phidias, who was said to have a secret to establish the proportions of his works that made them especially attractive to the eye and it was this proportion ϕ . Phidias' fame derived from the creation of the statue of Olympian Zeus, installed in the temple of Zeus at Olympia near Athens in the ancient Greece and considered one of the seven wonders of the ancient world.

The principle consisted of a proportion that was apparently considered whenever it was required to draw the design of the human body and this was extrapolated to all objects in the scene. It was also used by Leonardo Da Vinci. But this secret hovered among the scholars who collected these curiosities of mathematics. It was described as a number that squared equals the same number plus 1.

Euclid defines it in his work "The Elements" as "two numbers a and b are at golden ratio if and only if $a/b = (a + b)/a$, which also indicates that it cannot be written between the division of two integers, it is an irrational. That is to say:

$$\frac{a}{b} = \frac{a + b}{a}$$

$$\frac{a}{b} = \frac{(a + b) : b}{a : b}$$

If we collect the latter, we obtain:

$$\frac{a}{b} = \frac{a/b + 1}{a/b}$$

$$\phi = \frac{\phi + 1}{\phi}$$

Then, it is a number such that when squared it is equal to the same number plus 1:

$$\phi^2 = \phi + 1$$

This condition coincides with the quadratic equation: $x^2 - x - 1 = 0$, which we can solve with the general formula:

$$x = \frac{1 \pm \sqrt{5}}{2}$$

Now, if it is a geometric solution we should consider only the positive result: $x = \frac{1+\sqrt{5}}{2}$ number that we already showed as irrational. Moreover, the other solution of the equation is $\frac{1-\sqrt{5}}{2} = \frac{-1}{x}$ check it by rationalizing $\frac{-1}{\frac{1+\sqrt{5}}{2}}$, the latter could be a coincidence? Interestingly this number can be

expressed in a segment of the Euclidean plane, explained below in Geogebra. Note that this number ϕ , has an incredible reduced Euclidean representation, with continuous fractional form $[1; 1]$. Doesn't this representation seem idealized to you?

As if it were the definition of "unit" for Euclid's algorithm.

$$\phi \approx 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + 1}}}} = \frac{13}{8} = 1.625$$

Achieving this number with the compass is very simple. Use Geogebra.

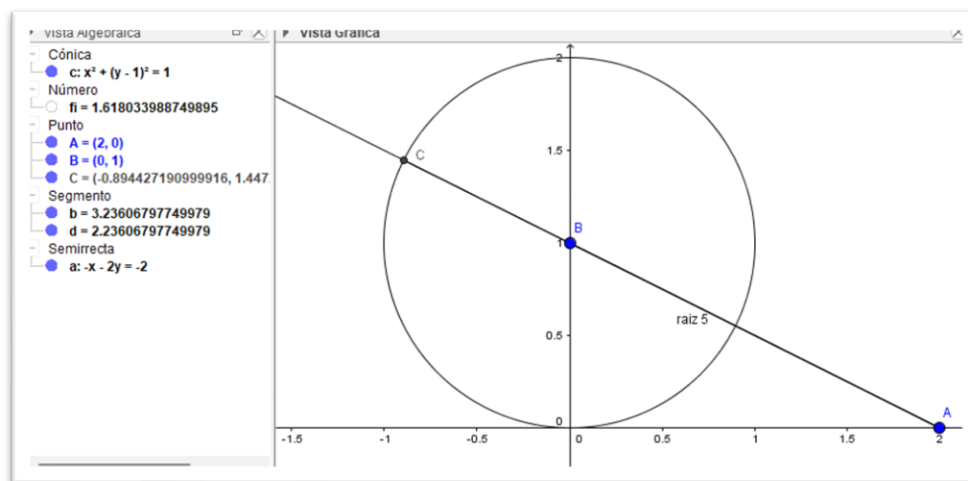


Figure 7. We obtain $\sqrt{5}$ in the hypotenuse of the triangle with legs 1 and 2.

Proceed by creating a right triangle, where the square $\sqrt{5}$ is located on the triangle's hypotenuse, and the legs measure 1 and 2. Then, subtract 1 from this hypotenuse segment. The excess length is exactly the number phi.

When the representation in Euclid's algorithm of this number ϕ became known, some mathematicians considered it pertinent to give them a name. Argentine mathematicians led by Vera G. de Spinadel (1929 -) in 1994, called them "metallic number family" [12], these are positive quadratic irrational numbers, which correspond to the positive solutions of the quadratic equation: $x^2 - bx - c = 0$ where both b and c are natural numbers. The number ϕ was called the golden number, of course the metallic number family is larger and its name has an explanation related to Euclid's algorithm.

At the time it was believed that this number could be the precursor of a new numeration and could be like the much sought after number pi, but clearly they were of different natures. The number ϕ was more related to a class of geometric and proportional numbers that were used as geometric series, in the context of arts and nature. With these quadratic equation solutions, we can explain the relationship it has with the Fibonacci series.

We have already said that: $\frac{F_{n+1}}{F_n} = \phi$ when n is very large, and by the related property of Fibonacci:

$$F_n + F_{n+1} = F_{n+2}$$

Then:

$$\frac{F_{n+2}}{F_{n+1}} = \frac{F_{n+1} + F_n}{F_{n+1}} = 1 + \frac{F_n}{F_{n+1}}$$

We collect the extremes of this equality:

$$\frac{F_{n+2}}{F_{n+1}} = 1 + \frac{1}{\frac{F_{n+1}}{F_n}}$$

Thus this will happen again for another n+1, and we are presented with a Euclidean reduction as n grows, in identical form of ϕ .

$$\frac{F_{n+3}}{F_{n+2}} = 1 + \frac{1}{1 + \frac{1}{\frac{F_{n+1}}{F_n}}} \quad \frac{F_{n+4}}{F_{n+3}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\frac{F_{n+1}}{F_n}}}}$$

In addition, the French mathematician Jacques Philippe Marie Binet (1786), defines the well-known Binet formula, which uses the two solutions of the quadratic equation already mentioned, and expresses something difficult to believe:

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

In this case, when n is very large we can eliminate part of the expression since $\lim_{n \rightarrow \infty} \left(\frac{1-\sqrt{5}}{2} \right)^n \rightarrow 0$

Other numbers were classified among the family of "metallic number family" [12], but none came from the numbers that motivate us, since the family was limited only to those that fulfilled the condition of being solutions of quadratic equations, which neither e or π do.

There are no official associations of metals with the number e (Euler) or π , but some mathematicians have proposed: The number e (Euler) is associated with platinum, because of its rarity and value in mathematics; the number π is associated with copper, because of its importance in geometry and circumference (or because it connects everything like electrical wiring).

7. Calculating numerical accuracy, relationship between old and new methods

The representation of decimal numbers in a computer is called floating point. This representation has a finite decimal point which under a robust numerical framework can be extended so that the computer considers more decimals. The usual numerical packages such as MatLab, SciLab, Mathematics; maintain a standard precision level of 15 decimal places, the rest are filled with zeros, depending on the number of operations required, since the numerical control algorithms maintain precision in iterated multiplications and divisions. The compensation of the decimal representation further away from the range is achieved with algebraic expressions such as fractions and Euclid's algorithm. So these structures are not dismissed out of hand, as if they were discontinued or obsolete. By no means, these old structures are used in large computational processes and operations. Today there are powerful tools available on the Internet, Wolfram Alpha: Computational Intelligence [11], which we will use to query numerical services without getting into the complications of representation.

Imagine that decimals can be stored in a vector of digits, so that they are treated as large numbers instead of tenths. The purpose will be to operate and update the results with long vectors of digits, where position is relevant.

The square root of 3 or sqrt(3) with the digital compass in Geogebra

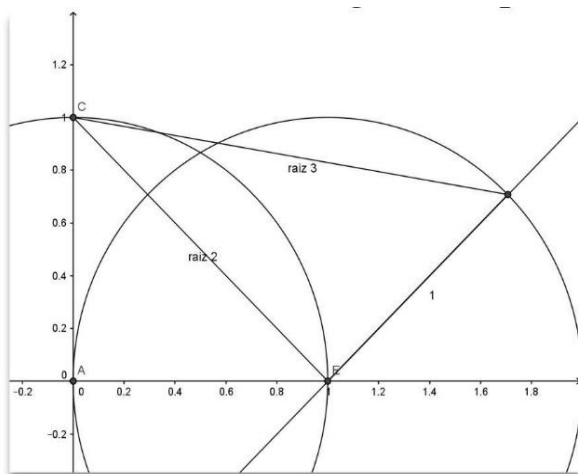


Figure 8. Sqrt(3) is obtained with a right triangle, with sqrt(2) and 1.

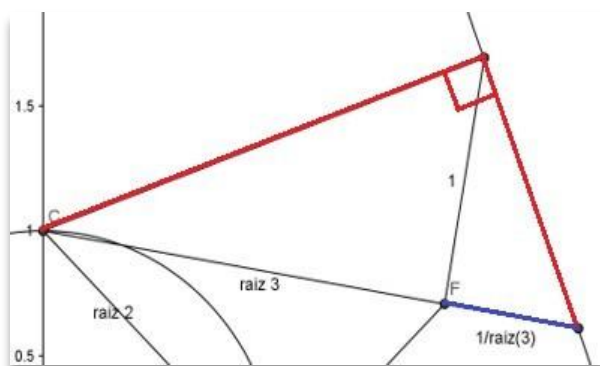


Figure 9. The reciprocal value of the sqrt(3) is obtained by extending the segment and applying Euclid's Theorem.

The sqrt(3) can be obtained in a segment with the Geogebra computer program.

We know that $\sqrt{2}$ corresponds to the diagonal of a square with sides 1. Therefore, if we build on this diagonal another right triangle with a leg on this diagonal and a leg of measure 1, the hypotenuse will be $\sqrt{3}$

Surely you already knew this geometric representation. Now we will use Euclid's theorem to obtain the reciprocal $\frac{1}{\sqrt{3}}$

I hope to surprise you a little with this idea. The segment $\sqrt{3}$ was used as the leg of a right triangle at F, the other leg of length 1.

Then another right triangle with red legs is completed (as shown in the figure). The prolongation of the segment where the root of 3 was, according to Euclid's Theorem corresponds to the reciprocal of the root of 3

If you weren't surprised, maybe with this one you will be. The reciprocal of root of 3 will be a number we will need later, and we require all its decimals to be considered by a large computer. We need, perhaps, more than 1000000 decimals of this number, so you may wonder how we could do this. If we proceed with Euclid's algorithm to approximate with a fraction in this segment we will systematically obtain:

$$\frac{1}{\sqrt{3}} \approx \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}} = 0.6$$

$$\frac{1}{\sqrt{3}} \approx \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + 1}}}}}}} = 0.5773195876$$

We do not need to continue any further, with these iterations we already have its continuous fractional expression.

Wolfram Alpha Computational Platform expresses it as:

$$\frac{1}{\sqrt{3}} \approx 0,57735026918962576450914878050195745564760175$$

$$\therefore 1270126876018602326483977672302933345693715395585749525225208713805135$$

This is in continuous fractional form $[0; 1, \overline{1, 2}]$

In this way and later, we will ask a computer to complete all the required decimals with Euclid's algorithm, doing operations of addition, multiplication and division, controlling in each case the projected decimal continuity. The idea is to do these operations as any schoolchild learning with 4 decimals, but now with the requested extension.

8. Leibniz's proposal to obtain the value of pi

Now with sums instead of products, it is the so-called Leibniz formula deduced in the 17th century by the German mathematician who gives his name to it. The process of Wallis integrals that we did earlier taught us to work with infinite sums, so you should not be surprised by Leibniz's proposal:

$$\pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots \right) \text{ or what is the same } \frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}$$

We can represent this in infinite sums, called alternating series due to their change of sign.

Functions associated with infinite sums.

Before, we can mention that there is a correspondence between some functions with infinite series. For example, we define as geometric series, the one that allows us to add the following numbers:

$$5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \frac{80}{81} + \frac{160}{243} + \frac{320}{729} = \sum_{k=1}^7 5 \cdot \left(\frac{2}{3}\right)^{k-1}$$

It is called geometric sum only because the regularity between the number succeeding the other is formed from the multiplication of $r = \frac{2}{3}$.

You can check that the result of this sum is obtained from the formula:

$$5 \cdot \frac{1 - \left(\frac{2}{3}\right)^7}{1 - \frac{2}{3}} = \frac{10295}{729} \approx 14.122$$

Moreover, this example considers 7 terms of the sum, but we can extend this sum to infinite since the multiplicative ratio has the property of being $r = \frac{2}{3} < 1$. That is $\lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^n = 0$, is satisfied, then:

$$\sum_{k=1}^{\infty} 5 \cdot \left(\frac{2}{3}\right)^{k-1} = 5 \cdot \frac{1 - \left(\frac{2}{3}\right)^{\infty}}{1 - \frac{2}{3}} = 5 \cdot \frac{1}{1 - \frac{2}{3}} = 15$$

Now if we make some changes, we come to the conclusion that infinite power functions "converge" whenever the value of r is between $-1 < r < 1$.

$$f(r) = 1 + r + r^2 + r^3 + \dots = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{as long as } |r| < 1$$

Thus, also, if we replace with $r = -x^2$, we get:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1 \cdot x^2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n} \quad \text{provided that } |x| < 1$$

Then, if we integrate the left and right sides, we obtain the so-called Gregory-Leibniz formula:

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$$

We check that if $x = 1$ is achieved $\arctan(1) = \frac{\pi}{4}$ and we obtain Leibniz's formula. However, $x = 1$ is not within the convergence interval of this series, that is why Leibniz develops other convergence criteria for alternating series. Then, if we change the beginning of $n=0$ to $k=1$, i.e. $n = k - 1$, we arrive at Leibniz's expression:

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2n+1} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}$$

9. Taylor series and Euler transforms

Taylor series approximations

Power series are infinite sums that approximate functions, the domain of the function is the interval of convergence of the series. Taylor (Frederick Winslow Taylor (1856-1915), industrial engineer by profession, was born in Philadelphia, USA). He defined the power series of any differentiable function f of order n , around $x = a$.

$$F(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{then } c_n = \frac{f^{(n)}(a)}{n!}$$

The advantages are enormous, since we can find a polynomial that approximates the function around a convergence interval where the value of $x = a$ is in the center. With this we will make a leap to the models of convergence of the number pi.

For example, if we use the function $f(x) = \arctan(x)$ and derive it to apply the Taylor series around $x = 0$, we obtain:

$$f(x) = \arctan(x); \quad f'(x) = \frac{1}{1+x^2}; \quad f''(x) = \frac{-2x}{(1+x^2)^2}; \quad f'''(x) = \frac{6x^2-2}{(1+x^2)^3};$$

$$f^{(4)}(x) = \frac{24x(1-x^2)}{(1+x^2)^4}; \quad f^{(5)}(x) = \frac{120x^4-240x^2+24}{(1+x^2)^5}$$

$$f(0) = 0; \quad f'(0) = 1; \quad f''(0) = 0; \quad f'''(0) = -2; \quad f^{(4)}(0) = 0; \quad f^{(5)}(0) = 24$$

Consider that when evaluating to zero the even derivatives also result in zero. Thus, we confirm the alternating series for the arctangent, but this time obtained by the Taylor series.

$$\arctan(x) = \frac{0}{0!}x^0 + \frac{1}{1!}x^1 + \frac{0}{2!}x^2 + \frac{-2}{2!3}x^3 + \frac{0}{4!}x^4 + \frac{24}{4!5}x^5 + \dots$$

So, then:

$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$$

This series relates to our theme because $\arctan(1) = \frac{\pi}{4}$

An example of Euler transforms with number series.

The Euler argument demonstrates that the harmonic series, which we know for those who have studied this, the harmonic is divergent and turns out to be convergent when it is alternating and furthermore converges to $\ln(2)$ (or the natural logarithm of 2).

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2)$$

Let us see how this is justified. The sum of terms of the alternating harmonic series can be written as an integral:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right) \Big|_0^1$$

We can think that it is: $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ and it is the result of an integral, then it is evaluated between 0 and

1 of the definite integral is obtained. But which integral are we talking about?

From this integral $\int_0^1 (1 - x + x^2 - x^3 + \dots) dx$. Do you agree?

We said earlier that:

$$1 + r + r^2 + r^3 + \dots = \frac{1}{1-r} \quad \text{provided that } |r| < 1$$

Thus, if we replace $r = -x$ we obtain:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1 \cdot x)^n = \sum_{n=0}^{\infty} (-1)^n \cdot x^n = 1 - x + x^2 - x^3 + \dots$$

Then:

$$\int_0^1 (1 - x + x^2 - x^3 + \dots) dx = \int_0^1 \frac{1}{1+x} dx = \ln(1+x) \Big|_0^1 = \ln(2)$$

Great, isn't it?

So, we can approximate numbers by series. Euler has more cards up his sleeve for these things.

Part 2. The secrets

The next part will be devoted to revealing the secrets to obtain an efficient algorithm to reach all decimals of pi. For various reasons these have not been fully revealed, and you may not know them. If you are a computer professional, you will be able to identify the techniques of large number processing. We will discuss integer operations on decimals to solve the problem of representing the floating point in a bounded environment, even if it is over the 100,000th decimal place or larger, if you prefer, of the number pi.

When we want to work with many decimals, we face the problem of numerical representation in computers, first let's go back to the Taylor series and then let's worry about the numerical representation. For all these models it is known, it is better to use fractions than decimal numbers, that is why the algorithms use mixed representations, thus, we only have to solve how to work with decimals that are far away from the floating point of their computational representation.

1. The most successful way with continuous function.

Let's try another arctangent.

The case is $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$. Then $6 \arctan\left(\frac{1}{\sqrt{3}}\right) = \pi$

This time the function will $f(x) = 6 \arctan\left(\frac{1}{\sqrt{3}} - x\right)$ to be evaluated at $x = 0$.

Deriving to obtain the Taylor series, we obtain:

$$6 \arctan\left(\frac{1}{\sqrt{3}} - x\right) = 6 \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{\sqrt{3}} - x\right)^{2n+1}}{2n+1}$$

for $\left|x - \frac{1}{\sqrt{3}}\right| < 1$

Then, the convergence interval is:

$$\begin{aligned} -1 < x - \frac{1}{\sqrt{3}} < 1 \\ -1 + \frac{1}{\sqrt{3}} < x < 1 + \frac{1}{\sqrt{3}} \\ -0.4226 < x < 1.577 \end{aligned}$$

That's fantastic; we've achieved it!

Then there is convergence for $x=0$.

Then, an excellent series would be:

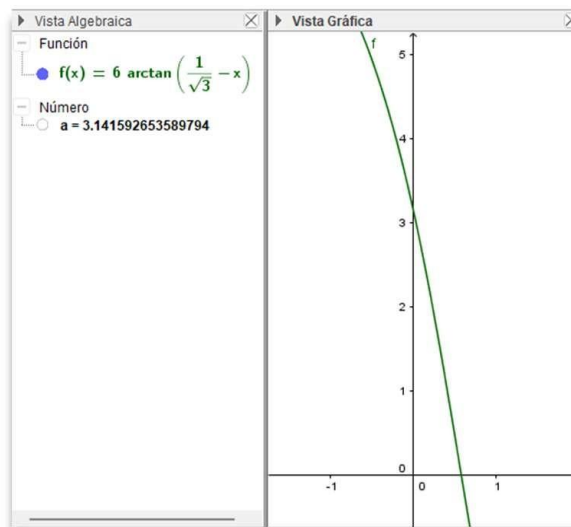


Figure 10. Decreasing curve of the arctangent.

$$6 \arctan \left(\frac{1}{\sqrt{3}} \right) = 6 \sum_{n=0}^{\infty} \frac{(-1)^n 3^{-\frac{2n+1}{2}}}{2n+1}$$

Then for $n=100$

$$6 \sum_{n=0}^{100} \frac{(-1)^n 3^{-\frac{2n+1}{2}}}{2n+1} = 3.1415926535897935$$

Thus, we discovered two very good series for our purpose. Both are convergent and systematically add real number terms progressively. The problem here is that, if we use an algorithm like this, we depend on the computer representation of decimals, because of these very small terms that must be added at the end of the series. That is, we cannot be sure that this is the correct result not because we are doing it wrong but because the computer uses a fixed number of decimals, about 15 decimals and when we must go on and affect those further decimals would be just those that are at the limit of the computer's numerical representation.

2. Newton's algorithm

In 1666 the mathematician and physicist Newton (Isaac Newton who proposed the theory of gravitational attraction) obtained a series to calculate the number pi and concluded it from the trigonometric value:

$$\sin \left(\frac{\pi}{6} \right) = \frac{1}{2} \Rightarrow \frac{\pi}{6} = \arcsin \left(\frac{1}{2} \right)$$

Studying Taylor's series, he finds the series of :

$$f(x) = \frac{\arcsin(2x)}{2x} = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2k+1} x^{2k} \quad \text{donde } f\left(\frac{1}{4}\right) = \frac{\pi/6}{1/2} = \frac{\pi}{3}$$

Then:

$$\sum_{k=0}^5 \frac{\binom{2k}{k}}{2k+1} \left(\frac{1}{4}\right)^{2k} \approx 1.047192239 \dots$$

Where the correct value of $\frac{\pi}{3} = 1.047197551 \dots$

Thus, increasing the n-value of the summation continued to approximate the remaining decimals [3]. We rate this algorithm as excellent.

3. The Bailey Borwein Plouffe (BBP) algorithm

It is a method that was developed in 1995 to calculate exclusively the number pi, with an infinite series, so that the terms can be calculated independently [15]. Simon Plouffe together with David Bailey and Peter Borwein obtained the formula using a computer program called PSLQ that searches for relations between integers and in base 16.

The formula they derived was:

Bailey Borwein Plouffe (BBP) algorithm

$$\pi = \sum_{k=0}^{\infty} \left(\frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right)$$

The five-in-five proof of the summation argument:

$$\begin{aligned} \sum_{k=0}^4 \left(\frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right) &= \frac{16071212445820879}{5115625817702400} \\ &\approx \mathbf{3,1415926}454603363195570212224423818317274066179799071866969806544... \\ \sum_{k=0}^9 \left(\frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right) &= \frac{64251934196540737654784844866951}{20452025861189303550405613977600} \\ &\approx 3,1415926\mathbf{5358979}11463887769659103474147790158884889967725870672423... \\ \sum_{k=0}^{14} \left(\frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right) &= \frac{517658978311277334141536972885988480280412713877204531}{164775970468280051996408614883427417773042630551142400} \\ &\approx 3.14159265358979\mathbf{323846}1732482037982486800056278143046732780578758091752793513404 \end{aligned}$$

The computational algorithm uses the numerical base 16 to optimize the calculation when many decimals are used, only then the result of the operation is expressed in decimal base.

4. The Chudnovsky brothers' algorithm

Led by one of the brothers Gregory Chudnovsky, they proposed an algorithm that is based on a Ramanujan formula. The Chudnovsky brothers (Ukrainians naturalized Americans) used their own algorithm to calculate 2.7 billion digits of pi in 2009 where they obtained a world record, and then progressively advanced until in 2011 they get 10 billion decimals of the number pi [13]. With this same algorithm, computer scientists Alexander Yee and Shigeru Kondo in 2013 manage to calculate 12.1 billion decimals [14].

The algorithm is based on the generalized hypergeometric series. This being alternating has Leibniz properties regarding its convergence.

$$\frac{1}{\pi} = 12 \cdot \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (13591409 + 545140134k)}{(3k)! (k!)^3 640320^{3k+3/2}}$$

The trick is to consider approximations by sections so that the result can be controlled in a vector of data that maintains the position of the decimals as if they were whole numbers, and thus to complete each segment where the sum term intervenes.

For example, if we observe the behavior of the algorithm on the first 50 known decimals of pi and from them validate a rule. Let us consider this number of correct decimals of pi.

$$\pi \approx 3.14159265358979323846264338327950288419716939937510582097494459230781640628620$$

8998628034 825342117068

And we use the algorithm:

Chudnovsky Brothers Algorithm

$$\pi = \frac{1}{12 \cdot \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (13591409 + 545140134k)}{(3k)! (k!)^3 640320^{3k+3/2}}}$$

If we calculate the first term of that sum, the one corresponding to $k = 0$, the approximation of pi obtained will be 1 divided by that result, which gives us the following (with sums from wolframalfa.com):

3.1415926535897342076684535915782983407622332609157

$$\frac{4\,268\,80\sqrt{10\,005}}{13\,591\,409} \approx 3.1415926535897342076684535915782983407622332609157$$

$$\therefore 06590894145498737666209401659108066117347469689758$$

Let us now calculate the first two terms. The approximation of pi will now be $k = 0, 1$ divided by the sum of them.

We get this:

3.1415926535897932384626433832795028841971676788548

$$\frac{27\,243\,597\,425\,235\,335\,774\,827\,985\,240\,064\,000\,000\sqrt{10\,005}}{867\,407\,410\,133\,324\,147\,761\,288\,805\,130\,794\,983\,129} \approx 3.1415926535897932384626$$

$$\therefore 43383279502884197167678854846287912727790370642977335176958726$$

As can be seen, the decimals that were already exact with the first term are maintained with this second term, and we add 14 more (they are the ones highlighted in bold). To do one more, let's see that the trend continues with the next term. By calculating 1 divided by the sum of the first three ($k = 0, 1, 2$) terms we get the following approximation of pi:

3.1415926535897932384626433832795028841971676788548

$$\frac{27\,243\,597\,425\,235\,335\,774\,827\,985\,240\,064\,000\,000\sqrt{10\,005}}{867\,407\,410\,133\,324\,147\,761\,288\,805\,130\,794\,983\,129} \approx 3.1415926535897932384626$$

$$\therefore 43383279502884197167678854846287912727790370642977335176958726$$

The previous ones are maintained, and 14 new exact decimals are added. And so on.

It is amazing to get sequences of 14 more exact decimal places with each additional term, since with very few terms we get a much closer approximation to the real value. Much better than any algorithm. So, we rate this algorithm as excellent.

5. Silva Selamé Algorithm (ASS)

If you find it incredible how they have been able to find and define these algorithms, now we will be the protagonists. According to the study we have done in this essay, we can think of joining all the considerations of the most outstanding mathematical authors in history and start from the beginning.

Let's develop our own algorithm, starting with:

$$6 \arctan \left(\frac{1}{\sqrt{3}} \right) = \pi$$

And we tried with: $f(x) = 6 \arctan \left(\frac{1}{\sqrt{3}} - x \right)$ for which arose from the power series Taylor in $x = 0$, $f(0) = \pi$.

$$\pi = 6 \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{\sqrt{3}} \right)^{2n+1}}{2n+1} = 6 \sum_{n=0}^{\infty} \frac{(-1)^n 3^{-\frac{2n+1}{2}}}{2n+1}$$

If we expand to $x \rightarrow 0$

$$\arctan \left(\frac{1}{\sqrt{3}} - x \right) = \frac{\pi}{6} - \frac{3x}{4} - \frac{3\sqrt{3}x^2}{16} + \frac{9\sqrt{3}x^4}{128} + \frac{27x^5}{320} + O(x^7)$$

With $x=0$

$$6 \arctan \left(\frac{1}{\sqrt{3}} \right) = 6 \sum_{n=0}^{\infty} \frac{(-1)^n 3^{-\frac{2n+1}{2}}}{2n+1}$$

Let's work on the sum of terms to get an expression of the algorithm

$$\begin{aligned} 6 \left(\sum_{n=0}^{11} \frac{(-1)^n 3^{-\frac{2n+1}{2}}}{2n+1} \right) &= 6 \left(\sum_{n=0}^{11} \frac{(-1)^n \left(\frac{\sqrt{3}}{3} \right)^{2n+1}}{2n+1} \right) = 6 \cdot \frac{\sqrt{3}}{3} \left(\sum_{n=0}^{11} \frac{(-1)^n \left(\frac{\sqrt{3}}{3} \right)^{2n}}{2n+1} \right) \\ &= 6 \cdot \frac{\sqrt{3}}{3} \left(\sum_{n=0}^{11} \frac{(-1)^n \left(\frac{1}{\sqrt{3}} \right)^{2n}}{2n+1} \right) \\ &= 2\sqrt{3} \left(\sum_{n=0}^{11} \frac{(-1)^n}{(2n+1)3^n} \right) = \mathbf{3.1415924542 \dots} \\ &= 2\sqrt{3} \left(\frac{1}{1} - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \frac{1}{729} - \frac{1}{2673} + \frac{1}{9477} - \frac{1}{32805} + \frac{1}{111537} - \frac{1}{373977} + \frac{1}{1240029} - \frac{1}{4074381} \right) \\ &= 2\sqrt{3} \left(\left(\frac{1}{1} - \frac{1}{9} \right) + \left(\frac{1}{45} - \frac{1}{189} \right) + \left(\frac{1}{729} - \frac{1}{2673} \right) + \left(\frac{1}{9477} - \frac{1}{32805} \right) + \left(\frac{1}{111537} - \frac{1}{373977} \right) + \left(\frac{1}{1240029} - \frac{1}{4074381} \right) \right) \\ &= 2\sqrt{3} \sum_{k=0}^5 \left(\frac{1}{(4k+1)3^{2k}} - \frac{1}{(4k+3)3^{2k+1}} \right) \\ &= 2\sqrt{3} \sum_{k=0}^5 \frac{1}{9^k} \left(\frac{1}{4k+1} - \frac{1}{12k+9} \right) \\ &= 2\sqrt{3} \left(\left[\frac{1}{9^0} \left(\frac{1}{1} - \frac{1}{9} \right) + \frac{1}{9^1} \left(\frac{1}{5} - \frac{1}{21} \right) \right] + \left[\frac{1}{9^2} \left(\frac{1}{9} - \frac{1}{33} \right) + \frac{1}{9^3} \left(\frac{1}{13} - \frac{1}{45} \right) \right] + \left[\frac{1}{9^4} \left(\frac{1}{17} - \frac{1}{57} \right) + \frac{1}{9^5} \left(\frac{1}{21} - \frac{1}{69} \right) \right] \right) \\ &= \mathbf{3.1415924542 \dots} \\ &= 2\sqrt{3} \sum_{n=0}^2 \left[\frac{1}{9^{2k}} \left(\frac{1}{8k+1} - \frac{1}{24k+9} \right) + \frac{1}{9^{2k+1}} \left(\frac{1}{8k+5} - \frac{1}{24k+21} \right) \right] \end{aligned}$$

Thus, we have a first version of our algorithm:

Silva Selamé Algorithm (ASS) $= 2\sqrt{3} \sum_{k=0}^{\infty} \frac{1}{81^k} \left[\frac{1}{8k+1} - \frac{1}{24k+9} + \frac{1}{72k+45} - \frac{1}{216k+189} \right]$
--

Let's try it five by five, as we did before with the BBP algorithm:

$$= 2\sqrt{3} \sum_{k=0}^4 \frac{1}{81^k} \left[\frac{1}{8k+1} - \frac{1}{24k+9} + \frac{1}{72k+45} - \frac{1}{216k+189} \right]$$

$$\frac{39109314160553059138384}{7187364992322595305225 \sqrt{3}}$$

$$\approx 3.1415926535714033817737105645779184574970837090255880006245033603911097486396735$$

$$\therefore 41872279003639093245$$

$$= 2\sqrt{3} \sum_{k=0}^9 \frac{1}{81^k} \left[\frac{1}{8k+1} - \frac{1}{24k+9} + \frac{1}{72k+45} - \frac{1}{216k+189} \right]$$

$$\frac{137966316378576542894866745012450339387116801633376}{25354938938138034138686381394493163496142563285825 \sqrt{3}}$$

$$\approx 3.14159265358979323845998904545815723164682333580898559851810755021711576515774234$$

$$\therefore 5078286000740054101$$

$$= 2\sqrt{3} \sum_{k=0}^{14} \frac{1}{81^k} \left[\frac{1}{8k+1} - \frac{1}{24k+9} + \frac{1}{72k+45} - \frac{1}{216k+189} \right]$$

$$\frac{14177725678419176989383216040837621457098441999820138432527910152984533089264}{2605529945958674055891953933258696968178141125973921415295979480405325356925 \sqrt{3}}$$

$$\approx 3.14159265358979323846264338327899429478611788675967126248193958028428440424653148$$

$$\therefore 7154654789614631463$$

We have compared these correct decimals of pi:

$$\pi \approx 3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628034825342117068$$

6. Computational Algorithm and Memory Management

Programming defines the memory spaces necessary to store all the decimals of the first calculations. It is important to start with a preset length, one billion decimals or more. From that moment on, the calculations can determine rational or irrational, and if all of them the total number of decimals will be considered. It begins then expressing those irrationals that are repeated in the process, will thus have the precision to continue the operations of addition and multiplication. The inverses can be expressed with all decimals and used as multiplication. So far so good if the calculations in the partial sections of the

algorithm do not result in decimals that are outside the possible ranges of the floating point computational representation.

For this reason, algorithms must consider computations on truncated decimal operations and treated as integers. Let us discuss the situation about an array of decimals of pi and the storage of surplus operations, when transiting from a specific point, in obtaining a set of decimals of pi.

3	1	4	1	5	9	2	6	5	3	5	8	9	7	9	3	2	3	8	4	5	9	9	8	9	0	4	5	4	5	8	1	5	7	2	3	.	.	.	.
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40

In an array (or memory vector) the previously obtained digits and the newly found digits are stored, these are located starting from the previous position and only from that point. The characteristics of the array can give clue of the exact place of the digits from where to continue.

Each set of calculations (summation argument) and with the current programming, can be distributed among different processors, 32 or 64 or more processors can synchronize the calculations and get very fast answers to obtain the billion or more decimal places of the number pi.

Truncated multiplication computes a truncated product, a contiguous subsequence of the digits of the product of 2 integers. Some truncated polynomial multiplication algorithms adapted to integers are presented. They are based on the most used full n-digit multiplication algorithms with a time complexity of $O(n^\alpha)$, with $1 < \alpha \leq 2$, but a constant 100 times faster [16]. For example, products of the least significant half with Karatsuba multiplication need only 80% of the full multiplication time. The faster the multiplication, the less relative time savings can be achieved [17].

7. Code implementation, ASS Algorithm in Python (gmpy2 library)

The Python language has a unique command library for numerical computation in a long-decimal execution environment. The library is called gmpy2 and allows declaring the number of precision decimals. Gmpy2 is an optimized Python extension module coded in C that supports fast multiple-precision arithmetic. It is based on the original gmpy module. **Gmpy2** adds support for true multiple-precision correctly rounded arithmetic (using the MPFR library) and complex arithmetic (using the MPC library). This means that, in particular, the most timeconsuming division operations are controlled and optimized for each command in the library.

Next, I will reveal one of the last secrets of these com algorithms. This algorithm we will code with this Python library and its execution will give answer about the decimals of pi to be counted from decimal n. Just as in Part 1 we learned how it was possible to obtain the Fibonacci number from a specific position, now the coding of the algorithm will give us a set of decimals of pi from the nth position of its decimals. This would allow us to avoid operating on the decimals already found and only solve for the decimals of pi in an environment that is relevant for our purposes.

Coding stages:

$$\sum_{k=0}^{\infty} \frac{2\sqrt{3}}{81^k} \left[\frac{1}{8k+1} - \frac{1}{24k+9} + \frac{1}{72k+45} - \frac{1}{216k+189} \right]$$

We will divide our algorithm into sections, for example, for the first section:

$$s1 = \sum_{k=0}^N \frac{2\sqrt{3}}{81^k} \left[\frac{1}{8k+1} \right]$$

Then to optimize an efficient decimal response, we modify this section starting from a value n and then N will be just a little higher, to optimize the precision around the decimal place n.

$$s1 = \left(\sum_{k=0}^n \left(\left(\frac{2\sqrt{3}}{81^k} \bmod [8k+1] \right) / [8k+1] \right) + \sum_{k=n+1}^N \frac{2\sqrt{3}}{81^k} \left[\frac{1}{8k+1} \right] \right)$$

The mod command provides only the integer remainder of the division, then this remainder is divided by [8k+1], thus taking care only of the remaining decimal places counting from position n. Then the sum up to N is added, to ensure precision around decimal place n.

The same is done with the other sums of the terms of the algorithm. For these, it will be necessary to eliminate the integer (before the comma) to control the result SumTotal= s1 - s2+ s3 - s4.

Below, we explain the complete code, which you can run in Google's Colab.

Complete code and execution

Let's go back to the digits of pi. The first case in red, counting from decimal 101 with 17 decimal places of precision. But then, the second case against decimal 901, with more 76 decimal places of precision. Both with 1000 decimal places of precision in the numerical context. Context, GoogleColab memory, no GPU.

```
n=900
```

```
gmpy2.get_context().precision=100030
```

- Digits from position 901
- It took less than 2 seconds to obtain the decimal places from 901

Definitive Python code

```
import sys
!{sys.executable} -m pip install gmpy2
!pip install gmpy2 --user # Install the gmpy2
import numpy as np
import math
import gmpy2
n=900
gmpy2.get_context().precision=100030
N=n+4
digitsPi=gmpy2.mpfr(0)
c=2*gmpy2.sqrt(3) # algorithm constants
lug=pow(10,n-1) # decimal position powers
cc=gmpy2.mpfr(c*lug) # position constant calculation
# first processor
pp=[] # list of partial sum calculations
```

```

Suma1a=gmpy2.mpfr(0) #start addition of remainder/divisor
for k in range(0,n+1):
    pp.append(gmpy2.div(gmpy2.mpfr(cc),gmpy2.mpfr(81**k))) #position precision
    divisor1 = gmpy2.mpz(8 * k + 1)
    resto1 = gmpy2.fmod(pp[k],gmpy2.mpfr(divisor1)) #MOD remainder, division whole
    Suma1a = Suma1a + gmpy2.div(gmpy2.mpfr(resto1),gmpy2.mpfr(divisor1)) # save division
Suma1b=gmpy2.mpfr(0) #start additional decimals
for k in range(n+1, N):
    pp.append(gmpy2.div(gmpy2.mpfr(cc),gmpy2.mpfr(81**k))) #position precision
    Suma1b = Suma1b + gmpy2.div(gmpy2.mpfr(pp[k]),gmpy2.mpfr(8 * k + 1))
Suma1 = Suma1a + Suma1b
Suma1 = Suma1 - gmpy2.trunc(Suma1) + 10.0 #Wildcard added for first priority
# second independent processor
Suma2a=gmpy2.mpfr(0) #start addition of remainder/divisor
for k in range(0,n+1):
    divisor2 = gmpy2.mpz(24 * k + 9)
    resto2 = gmpy2.fmod(pp[k],gmpy2.mpfr(divisor2)) #MOD remainder, division whole
    Suma2a = Suma2a + gmpy2.div(gmpy2.mpfr(resto2),gmpy2.mpfr(divisor2)) # save division
Suma2b=gmpy2.mpfr(0) # start additional decimals
for k in range(n+1, N):
    Suma2b=Suma2b+gmpy2.div(gmpy2.mpfr(pp[k]),gmpy2.mpfr(24 * k + 9))
Suma2 = Suma2a + Suma2b
Suma2 = Suma2 - gmpy2.trunc(Suma2) + 5.0 #Wildcard added for first priority in operation
# third independent processor
Suma3a = gmpy2.mpfr(0) #start addition of remainder/divisor
for k in range(0,n+1):
    divisor3 = gmpy2.mpz(72 * k + 45)
    resto3 = gmpy2.fmod(pp[k],gmpy2.mpfr(divisor3)) #MOD remainder, division
    Suma3a = Suma3a + gmpy2.div(gmpy2.mpfr(resto3),gmpy2.mpfr(divisor3))
Suma3b=gmpy2.mpfr(0)
for k in range(n+1, N):
    Suma3b = Suma3b + gmpy2.div(gmpy2.mpfr(pp[k]),gmpy2.mpfr(72 * k + 45))
Suma3 = Suma3a + Suma3b
Suma3 = Suma3 - gmpy2.trunc(Suma3) + 2.0 #Wildcard
# fourth independent processor
Suma4a=gmpy2.mpfr(0)
for k in range(0,n+1):
    divisor4 = gmpy2.mpz(216 * k + 189)
    resto4 = gmpy2.fmod(pp[k],gmpy2.mpfr(divisor4))
    Suma4a = Suma4a + gmpy2.div(gmpy2.mpfr(resto4),gmpy2.mpfr(divisor4))
Suma4b = gmpy2.mpfr(0)
for k in range(n+1, N):
    Suma4b = Suma4b + gmpy2.div(gmpy2.mpfr(pp[k]),gmpy2.mpfr(216 * k + 189))
Suma4 = Suma4a + Suma4b
Suma4 = Suma4 - gmpy2.trunc(Suma4)
#Return to the first processor
#The four processes are added to obtain the digits from n+1

```

```
digitsPi = Suma1 - Suma2 + Suma3 - Suma4
digitsPi = digitsPi - gmpy2.trunc(digitsPi) #removing wildcards from operation
print("\n Digits for Pi=",10000000000000000000*digitsPi)
```

Context, GoogleColab memory, no GPU

n=100000

gmpy2.get_context().precision=100030

- The digits from position 100001
- It took 48 minutes to get the decimals from the decimal of pi, or to count from 100001

35313143701227785.9945834190409522277028782321011429272630984990069538973640900
7078663919526994412320084652362036352279720143549310285770719666221

8. Conclusion

This paper was an essay that brought together the main motivations of mathematicians in history that lead to this enigmatic number pi. We detail with simplicity the mathematical representations of researchers, from different eras, to answer the original problem of squaring the area of the unit circle, equal to pi.

The contribution of the quadrature of Hippocrates' lunula, highlighted by the mathematician Alhacen, raised the expectations at the time for solving the squaring of the circle. We learned how continuous functions can graph the complex scheme leading to this number. The intervention of the mathematician Euler, which radically changed the scenario, managed to demonstrate that complex numbers could be a better option to understand the original problem. Euler reveals the relationship between the two irrational numbers: $e = 2,718281\cdots$ and $\pi = 3,141592 \cdots$. Nowadays, the convergence techniques of alternating series allowed computer technology to define algorithms that can systematically approximate all decimals of the number pi. We highlight the series method to find our own ASS algorithm (authored by this author) for decimal approximation

We conclude that, with the mathematical exercise of the series, a simple way to obtain computational algorithms to find decimals for this copper number pi, or for any other number, is completed. We were able to help Akira Haraguchi recite the digits of pi and continue with them from 100,001 onwards.

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